Risk and Ambiguity

The Arrow-Pratt Premium

✓ coefficient of absolute risk aversion (CARA): $A(w) = -\frac{U''(w)}{U'(w)}$

✓ coefficient of relative risk aversion (CRRA): $\mathrm{R}(w) = w\mathrm{A}(w) = -w\frac{U''(w)}{U'(w)}$

- W = current wealth
- z = random gamble payoffs where

  $E(z) = 0, Var(z) = \sigma_z^2$
- $W + z$ = wealth given gamble
- $\pi_A = \pi(W, z)$ = absolute risk Premium
- $\pi_R = \frac{\pi_A}{\mathrm{E(W+z)}}$ = relative risk premium

- Certainty-Equivalent Value $CE(W+z) = U^{-1}(E[U(W+z)])$
- The absolute risk premium is defined by

$$\pi_A = \pi(W, z) = E(W+z) - CE(W+z) = W + E(z) - CE(W+z)$$

$$CE(W+z) = W + E(z) - \pi_A$$

$$\xrightarrow{U(\cdot)} E[U(W+z)] = U[W + E(z) - \pi_A] = U[W - \pi_A]$$

LHS: expected utility of the current level of wealth, given the gamble

RHS: utility of the current level of wealth plus the expected value of the gamble less the risk premium

- The relative risk premium: $\pi_R = \frac{\pi_A}{\mathrm{E(W+z)}} = 1 - \frac{CE(W+z)}{E(W+z)}$, $CE(W+z) = E(W+z) \times (1 - \pi_R)$

$$\xrightarrow{U(\cdot)} E[U(W+z)] = U[E(W+z) \times (1 - \pi_R)] = U[W(1 - \pi_\mathrm{R})] = U[W - W\pi_R]$$

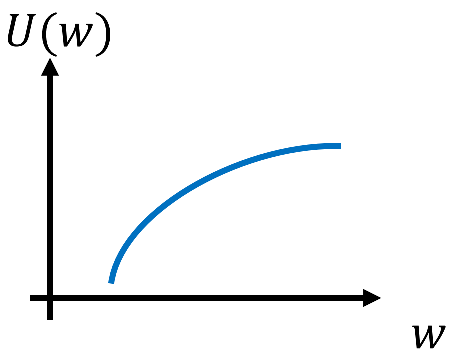
- $E[U(W+z)] = U[W-\pi_A]$
  $= U[W - W\pi_R]$
- By Taylor series expansion (around W)
  - CARA:
    ✓ LHS $= E\left[U(W) + zU'(W) + \frac{1}{2}z^2U''(W)\right] = U(W) + \frac{1}{2}Var(z)U''(W)$
    ✓ RHS $= U(W) - \pi_A U'(W)$ (Pratt assumes that second order and higher terms are insignificant)
    → $U(W) + \frac{1}{2}Var(z)U''(W) = U(W) - \pi_A U'(W)$,
    $$\pi_A = \frac{1}{2}\left(-\frac{U''(W)}{U'(W)}\right)Var(z) = \frac{1}{2}\left(-\frac{U''(E(W+z))}{U'(E(W+z))}\right)Var(W+z)$$
    → $A(w) = -\frac{U''(w)}{U'(w)}$
  - CRRA:
    ✓ LHS $= U(W) + \frac{1}{2}Var(z)U''(W)$
    ✓ RHS $= U(W) - W\pi_R U'(W)$
    → $U(W) + \frac{1}{2}Var(z)U''(W) = U(W) - W\pi_R U'(W)$,
    $$\pi_R = \frac{1}{2}\left(-\frac{U''(W)}{U'(W)}\right)\frac{Var(z)}{W} = \frac{1}{2}\left(-\frac{U''(W)}{U'(W)}\right)\frac{Var(W+z)}{E(W+z)}$$
    $$= \frac{1}{2}\left(-\frac{U''(W)}{U'(W)}\right)E(W+z)\frac{Var(W+z)}{(E(W+z))^2} = \frac{1}{2}\left(-\frac{U''(E(W+z))}{U'(E(W+z))}\right)E(W+z)Var\left(\frac{W+z}{E(W+z)}\right)$$
    → $\mathrm{R(w)} = -w\frac{U''(w)}{U'(w)}$

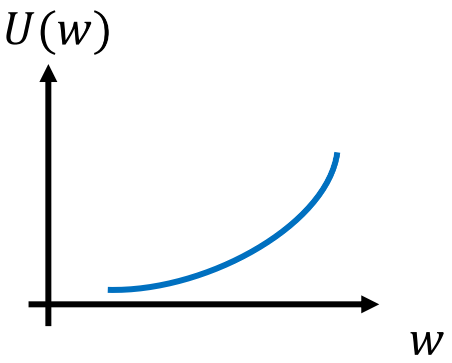
$$A(w) = -\frac{U''(w)}{U'(w)}, \pi_A = \frac{1}{2}A(w)Var(w)$$

$$\frac{dA(w)}{dw} = \frac{-U'''(w)U'(w) + \left(U''(w)\right)^2}{\left(U'(w)\right)^2}$$

$$\frac{dA(w)}{dw} > 0 \Rightarrow U'''(w) < 0$$

$$\frac{dA(w)}{dw} < 0 \Rightarrow U'''(w) > 0$$





$$R(w) = -w\frac{U''(w)}{U'(w)}, \pi_R = \frac{1}{2}R(w)Var\left(\frac{w}{E(w)}\right)$$

$$\frac{dR(w)}{dw} = \frac{-\left(U''(w) + wU'''(w)\right)U'(w) + w\left(U''(w)\right)^2}{\left(U'(w)\right)^2}$$

$$\frac{dR(w)}{dw} > 0 \Rightarrow \left(U''(w) + wU'''(w)\right) < 0$$

$$\frac{dR(w)}{dw} < 0 \Rightarrow \left(U''(w) + wU'''(w)\right) > 0$$

- CARA: $A(w) = -\frac{U''(w)}{U'(w)}$

  1. 恆定型絕對風險趨避（Constant Absolute Risk Aversion, CARA）：對於風險的趨避程度不取決於資產的多少，即使資產增加，對風險的趨避不變，最高投資數額不變，即 $\frac{dA(w)}{dw} = 0$，則可以稱作恆定型絕對風險趨避。
  2. 遞減型絕對風險趨避（Decreasing Absolute Risk Aversion, DARA）:隨著資產的增加，對於風險的趨避程度降低，最高投資數額變大，即 $\frac{dA(w)}{dw} < 0$，則可以稱作遞減型絕對風險趨避。
  3. 遞增型絕對風險趨避（Increasing Absolute Risk Aversion, IARA）：隨著資產的增加，對於風險的趨避程度增加，最高投資數額變小，即 $\frac{dA(w)}{dw} > 0$ ，則可以稱作遞增型絕對風險趨避。

- CRRA: $\mathrm{R}(w) = -w\frac{U''(w)}{U'(w)}$

  1. 恆定型相對風險趨避（Constant Relative Risk Aversion, CRRA）：投資數額占總資產的比率不隨總資產的變化而變化，無論總資產增加或減少，投資數額都占固定的比率（比如10%），如果 $\frac{dR(w)}{dw} = 0$成立，則可定義為恆定型相對風險趨避。
  2. 遞減型相對風險趨避（Decreasing Relative Risk Aversion, DRRA）：投資數額占總資產的比率隨總資產的增加而增加,表示對風險的趨避程度降低，如果$\frac{dR(w)}{dw} < 0$成立，則可定義為遞減型相對風險趨避。
  3. 遞增型相對風險趨避（Increasing Relative Risk Aversion, IRRA）：投資數額占總資產的比率隨總資產的增加而減少,表示對風險的趨避程度增加，如果 $\frac{dR(w)}{dw} > 0$ 成立，則可定義為遞增型相對風險趨避。

資料來源：維基百科

- Arrow-Pratt’s risk theory: CARA、CRRA
- example: our utility function is $U(w) = e^{-\beta \mathrm{t}} W^{\gamma}$
- CARA: $\mathrm{A(W)} = -\frac{U''(W)}{U'(W)} = -\frac{e^{-\beta t}\gamma(\gamma-1)W^{\gamma-2}}{e^{-\beta t}\gamma W^{\gamma-1}} = \frac{1-\gamma}{W}$, $\frac{dA(w)}{dw} = -\frac{1-\gamma}{W^2} < 0$ → 遞減型絕對風險趨避
- CRRA: $\mathrm{R(W)} = -W\frac{U''(W)}{U'(W)} = -W\frac{e^{-\beta t}\gamma(\gamma-1)W^{\gamma-2}}{e^{-\beta t}\gamma W^{\gamma-1}} = 1-\gamma$, $\frac{dR(w)}{dw} = 0$ → 恆定型相對風險趨避
- $1-\gamma$愈大(γ愈小)，愈風險趨避

- $U(W) = \frac{w^{\gamma}}{1-\gamma}$
- CARA: $\mathrm{A(W)} = -\frac{U''(W)}{U'(W)} = -\frac{-\gamma W^{\gamma-2}}{\gamma\frac{W^{\gamma-1}}{1-\gamma}} = \frac{1-\gamma}{W}$
- CRRA: $\mathrm{R(W)} = -W\frac{U''(W)}{U'(W)} = 1-\gamma$

Ambiguity

- The use of the term “ambiguity” to describe a particular type of uncertainty is due to Daniel Ellsberg in his classic 1961 article and 1962 PhD thesis, who informally described it as:

> “the nature of one’s information concerning the relative likelihood of events… a quality depending on the amount, type, reliability and ‘unanimity’ of information, and giving rise to one’s degree of ‘confidence’ in an estimation of relative likelihoods.” (1961,p.657)

- Unlike the economic concepts of “risk” and “risk aversion,” there is not unanimous agreement on what “ambiguity aversion,” or even “ambiguity” itself, exactly is. However several models and definitions have been proposed.

- state space $\mathcal{S}$ with a common partition $\{E_1, \dots, E_n\}$
- Preferences are defined over the domain of horse-roulette acts – henceforth called acts – namely maps $f = (\dots; P_j\ if\ E_j; \dots) = (\dots; (\dots; x_{ij}, p_{ij}; \dots), E_j; \dots)$ from a (finite or infinite) state space $\mathcal{S}$ to roulette lotteries P_j over a set of prizes $\mathcal{X}$.
- acts $f = \{\dots; \boldsymbol{P}_j\ if\ E_j; \dots\}$ and $g = \{\dots; \boldsymbol{Q}_j\ if\ E_j; \dots\}$
- given probability $\alpha \in (0,1)$, the mixture $\alpha \cdot f + (1-\alpha) \cdot g$ is defined as the act

$$\alpha \cdot f + (1-\alpha) \cdot g = \{\dots; \alpha \boldsymbol{P}_j + (1-\alpha)\boldsymbol{Q}_j; \dots\}$$

- Axioms

1. Weak order: $\forall\ f, g, h \in \mathcal{F}$ (1) either $f \succcurlyeq g$ or g $\succcurlyeq f$ (2) if $f \succcurlyeq g$ and $g \succcurlyeq h$, then $f \succcurlyeq h$
2. Non-Degeneracy: There exists acts f and g for which $f \succ g$.
3. Continuity: $\forall$ acts f, g, h, if $f \succ g$ and $g \succ h$, there exists $\alpha, \beta \in (0,1)$ such that

$$\alpha \cdot f + (1-\alpha) \cdot h \succ g \text{ and g} \succ \beta \cdot f + (1-\beta) \cdot h$$

4. Independence: $\forall$ acts f, g, h and all $\alpha \in (0,1)$,

$$f \succcurlyeq g \Leftrightarrow \alpha \cdot f + (1-\alpha) \cdot h \succcurlyeq \alpha \cdot g + (1-\alpha) \cdot h$$

5. Monotonicity: $\forall$ acts f, g, if the roulette lottery $f(s)$ is weakly preferred to the roulette lottery $g(s)$ for every state s, then $f \succcurlyeq g$

- Maxmin Expected Utility (MEU, or called the Multiple-Priors(MP) model) (Daniel Ellsberg,1961)
  - Consider a closed, convex set C of probability measures – priors – on the state space $\mathcal{S}$, a von Neumann-Morgenstern utility function $U(\cdot)$
  - The expected utility of preference over act $f(\cdot)$ is evaluated as $W\big(f(\cdot)\big) = \min_{\mu \in C} \int U\big(f(\cdot)\big) d\mu$

- e.g. As Ellsberg's primary examples, he offered two thought-experiment decision problems, which remain the primary motivating factors of research on ambiguity and ambiguity aversion to the present day. The most frequently cited of these, known as the Three-Color Ellsberg Paradox.

$$W(f(\cdot)) = \min_{\mu \in C} \int U(f(\cdot))d\mu$$

THREE-COLOR ELLSBERG PARADOX (single urn)	30 balls	60 balls	
	red	black	yellow
a_1	\$100	\$0	\$0
a_2	\$0	\$100	\$0
a_3	\$100	\$0	\$100
a_4	\$0	\$100	\$100

  - Let the state space be $\{s_r, s_b, s_y\}$, where s_r denotes the draw of a red ball, etc.
  - Let the set of prizes be $\mathcal{X} = \{\$0,\ \$100\}$
  - Set $U(\$100) = 1$ and $U(\$0) = 0$
  - To reflect the assumption that 30 out of the 90 balls in the urn are red, but that the number of black and yellow balls is not known, consider the set of priors $C = \left\{\mu \in \Delta(\mathcal{S}) : \mu(s_r) = \frac{1}{3}\right\}$
  - Every prior $\mu \in C$ assigns probability $\frac{1}{3}$ to the state s_r → $W(a_1) = \frac{1}{3}$
  - Every prior $\mu \in C$ assigns probability $\frac{2}{3}$ to the state $\{s_b, s_y\}$ → $W(a_4) = \frac{2}{3}$
  - Act a_2 yields \$100 on state s_b and \$0 otherwise → it is a bet on black.
    The prior in C assigns $P(s_b) = 0, P(s_y) = \frac{2}{3}$ such that minimized expected utility. → $W(a_2) = 0$
  - Act a_3 yields \$100 on the event $\{s_r, s_y\}$ and zero otherwise → it is a bet against black.
    The prior in C assigns $P(s_b) = \frac{2}{3}, P(s_y) = 0$ such that minimizes expected utility. → $W(a_3) = \frac{1}{3}$

➔ $a_1 \succ a_2$ and $a_3 \prec a_4$

- $W(f) = \int_{\mathcal{S}} U\big(f(s)\big)d\mu(s) = \sum_{j=1}^{n} U\big(\boldsymbol{P}_j\big) \cdot \mu\big(E_j\big) = \sum_{j=1}^{n}\big[\sum_i U\big(x_{ij}\big)p_{ij}\big] \cdot \mu\big(E_j\big)$

- $W\big(f(\cdot)\big) = \rho \cdot \int U\big(f(\cdot)\big)d\mu_0 + (1-\rho) \cdot \min_{\mu \in D} \int U\big(f(\cdot)\big)d\mu$ (Daniel Ellsberg,1961)
  - $\rho \in (0,1)$ represents the individual's "degree of confidence" in the estimate μ_0, D is a set of distributions that still seem 'reasonable,' and $C = \rho \cdot \mu_0 + (1-\rho) \cdot D$ is seen to be the set of priors.

- $W\big(f(\cdot)\big) = \alpha \cdot \min_{\mu \in C} \int U\big(f(\cdot)\big)d\mu + (1-\alpha) \cdot \max_{\mu \in C} \int U\big(f(\cdot)\big)d\mu$ (Gilboa and Schmeidler, 1989)
  - Called α-maxmin, or α-MEU model
  - For $\alpha = 1$, this representation reduces to MEU, and for $\alpha = 0$ it reduces to what is termed max*max* expected utility max $\int_C U\big(f(s)\big)d\mu$, and it allows for a whole range of intermediate attitudes toward ambiguity.

- Expected utility with uncertain probabilities (EUUP) (Izhakian, 2017)
  - $V(X) = \int_{z \le 0}\big[1 - \gamma^{-1}\big(\int_{\mathcal{P}} \gamma\big(P(U(X) \ge z)\big)d\xi\big)\big]dz + \int_{z \ge 0} \gamma^{-1}\big(\int_{\mathcal{P}} \gamma\big(P(U(X) \ge z)\big)d\xi\big)dz$
  - X is the investment payoff, $\gamma: [0,1] \to \mathbb{R}$ is strictly increasing and twice-differentiable

- $\mathcal{S}$ be a state space
- f is an act on a state space $\mathcal{S}$
- ξ is a probability measure on an algebra of subset of $\mathcal{P}$
- $E = \{t \in \mathcal{S} | f(t) \geq f(s)\}$

- Ambiguity – the uncertainty about probabilities – plays a role in the probability formation phase, while risk – the uncertainty about consequences – plays a role in the valuation phase.
- Similarly to Arrow-Pratt's risk theory, the coefficient of absolute ambiguity aversion (CAAA) can be defined by $-\frac{\gamma''(P(E))}{\gamma'(P(E))}$, and the coefficient of relative ambiguity aversion (CRAA) by $-\frac{\gamma''(P(E))}{\gamma'(P(E))}P(E)$

2. Invariant biseparable preferences

In this section, we introduce the basic preference model that is used throughout the paper, and show that it generalizes all the popular models of ambiguity-sensitive preferences.

The model is characterized by the following five axioms:

Axiom 1 (Weak order). *For all* $f, g, h \in \mathcal{F}$: (1) *either* $f \succsim g$ *or* $g \succsim f$, (2) *if* $f \succsim g$ *and* $g \succsim h$, *then* $f \succsim h$.

Axiom 2 (Certainty independence). *If* $f, g \in \mathcal{F}$, $x \in X$, *and* $\lambda \in (0, 1]$, *then*

$$f \succsim g \Leftrightarrow \lambda f + (1 - \lambda)x \succsim \lambda g + (1 - \lambda)x.$$

Axiom 3 (Archimedean axiom). *If* $f, g, h \in \mathcal{F}$, $f \succ g$, *and* $g \succ h$, *then there exist* $\lambda, \mu \in (0, 1)$ *such that*

$$\lambda f + (1 - \lambda)h \succ g \quad \text{and} \quad g \succ \mu f + (1 - \mu)h.$$

Axiom 4 (Monotonicity). *If* $f, g \in \mathcal{F}$ *and* $f(s) \succsim g(s)$ *for all* $s \in S$, *then* $f \succsim g$.

Axiom 5 (Nondegeneracy). *There are* $f, g \in \mathcal{F}$ *such that* $f \succ g$.